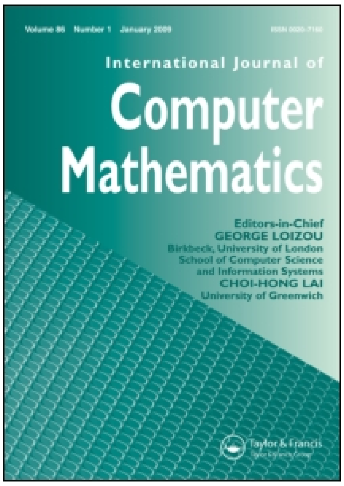

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On the numerical solution of the Burgers's equation

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On the numerical solution of the Burgers's equation

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This paper is dedicated to professor Faezeh Toutounian on the occasion of her 60th birthday

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In this paper, we consider the linear heat equation arisen from the Burgers's equation using the Hopf–Cole transformation. Discretization of this equation with respect to the space variable results in a linear system of ordinary differential equations. The solution of this system involves in computing $\exp(\alpha A)y$ for some vector y, where A is a large special tridiagonal matrix and α is a positive real number. We give an explicit expression for computing $\exp(\alpha A)y$. Finally, some numerical experiments are given to show the efficiency of the method.

Keywords: Burgers' equation; Hopf–Cole transformation; semi-discretization; system of ODEs; matrix exponential

2000 AMS Subject Classification: 35K55, 15A15

1. Introduction

Consider the Burgers's equation

$$u_t + uu_x = \nu u_{xx} \quad 0 < x < \ell, \ \ t > 0, \tag{1}$$

subject to the initial and boundary conditions

$$\begin{aligned} u(x,0) &= f(x), \quad 0 \le x \le \ell, \\ u(0,t) &= g_0(t), \quad t \ge 0, \\ u(\ell,t) &= g_1(t), \quad t \ge 0. \end{aligned}$$

The Burgers's equation appears in many areas of engineering sciences such as models of traffic, turbulence and fluid flow. There are several methods for solving the Burgers's equation. Some of them seek an exact solution of it. These methods are generally classified into two subclasses. In the first class, by using the wave transformation $u(x,t) = U(\xi)$, where $\xi = x - \nu t$, the Burgers's

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equation is changed into a nonlinear ODE involving U and its first derivative. Then a solution of the form

$$U(\xi) = \sum_{i=0}^{n} a_i \varphi^i(\xi),$$

with $\varphi(\xi) = \tanh\xi, \operatorname{sech}\xi, \sec\xi, \operatorname{sn}\xi, \operatorname{cn}\xi$, etc. is sought. It is necessary to mention that $\operatorname{sn}\xi$ and $\operatorname{cn}\xi$ are the Jacobi elliptic sine and cosine functions, respectively. Some of the methods in this category are the tanh-function method [18,21,22,25], the modified extended tanh-function method [4,5,27] and the Jacobi elliptic function expansion method [7,19]. In the second class, the exact solutions are sought by using the following transformation

$$u(x,t) = F(\varphi_1, \varphi_2, \ldots), \quad G(\varphi_1, \varphi_2, \ldots) = 0.$$

After determining the functions φ_i from $G(\varphi_1, \varphi_2, \ldots) = 0$, one can obtain the exact solution u via $u(x,t) = F(\varphi_1, \varphi_2, \ldots)$. Some of the methods in this category are the Bäcklund transformation [6,14,20,29], Darboux transformation [23] and sine–cosine methods [30–33]. In another method in this category, the equation is converted to a heat equation by the Hopf–Cole transformation [12] and then the solution of this heat equation is obtained. Taking different methods for solving this heat equation gives various methods for solving the Burgers's equation.

There are many ways to obtain an approximate solution of the Burgers's equation. In [8] the Adomian decomposition is applied for solving the Burgers's equation. A finite-difference approach can be found in [28]. Mixed finite-difference and boundary element methods were proposed in [1]. In [16], Kutluay et al. have presented explicit and exact-explicit finite difference methods for solving Equation (1). They have discretized the heat equation arisen from Hopf–Cole transformation and given an explicit expression for the exact solution of it. In [17], the authors have obtained an approximate solution of the equation by the least-squares quadratic B-spline finite element method. In [15], the numerical results obtained using a lumped Galerkin method with quadratic B-spline finite elements have been given. Caldwell and Smith in [3] and Caldwell et al. in [2] proposed the finite element and cubic spline finite element methods, respectively, for computing an approximate solution of Equation (1). Another method based on the direct discretization of Equation (1) was proposed by Hon et al. in [11]. Another approach is to find the solution of the heat equation arisen from Hopf-Cole transformation by its semi-discretization. In this case a matrix exponential should be computed. In [10], an approximation of this matrix exponential was obtained by the restrictive Taylor method [13]. In [9], the authors have computed this matrix exponential by a restrictive Pade approximation.

We focus our attention on the linear heat equation arisen from Equation (1) by the Hopf–Cole transformation and its semi-discretization. Then a method for computing the arisen matrix exponential is proposed. For this purpose, we consider two problems of [1] as follows.

Problem 1

$$\begin{aligned} u_t + uu_x &= \nu u_{xx}, \quad 0 < x < 1, \ \ t > 0, \\ u(x,0) &= \sin(\pi x), \quad 0 \le x \le 1, \\ u(0,t) &= u(1,t) = 0, \quad t \ge 0. \end{aligned}$$

Problem 2

$$\begin{aligned} u_t + uu_x &= \nu u_{xx}, \quad 0 < x < 1, \ \ t > 0, \\ u(x,0) &= 4x(1-x), \quad 0 \le x \le 1, \\ u(0,t) &= u(1,t) = 0, \quad t \ge 0. \end{aligned}$$

The series representation of the analytical solution of these problems can be found in [1]. By the Hopf–Cole transformation $u = -2\nu\theta_x/\theta$, these equations are transformed into

$$\theta_t = \nu\theta_{xx}, \tag{2}$$

with the initial conditions for Problems 1 and 2,

$$\theta(x,0) = \theta_0(x) = \exp\{-(2\pi\nu)^{-1}(1-\cos\pi x)\}, \quad 0 < x < 1,$$
$$\theta(x,0) = \theta_0(x) = \exp\{-(3-2x)x^2/3\nu\}, \quad 0 < x < 1,$$

respectively, and boundary conditions for both of the problems are

$$\theta_x(0,t) = \theta_x(1,t) = 0, \quad t > 0.$$

The semi-discretization of Equation (2) with respect to the space variable results in a linear system of ordinary differential equations. The solution of this system involves in computing $\exp(\alpha A)y$ for some vector y, where α is a positive real number and A is usually a large special tridiagonal matrix. In this paper, an explicit expression for computing $\exp(\alpha A)y$ is given.

This paper is organized as follows. In Section 2, the explicit expression for $\exp(\alpha A)y$ is obtained. Section 3 is devoted to some numerical experiments. Concluding remarks are also given in Section 4.

2. Main results

We subdivide the interval $0 \le x \le 1$ into $n+1$ equal subintervals by the grid points $x_i = ih$, $i = 0, 1, 2, \ldots, n+1$, where $h = 1/(n+1)$ and write down Equation (2) at every mesh point $x_i = ih,\ i = 1, 2, \ldots, n$, along time level t. Then, we substitute

$$\frac{\theta(x_{i+1},t) - 2\theta(x_i,t) + \theta(x_{i-1},t)}{h^2} + O(h^2),$$

for $\theta_{xx}(x_i,t)$ in Equation (2). Now, we denote $\theta(x_j,t)$ by $\theta_j(t)$. Neglecting the terms of order h^2, it then follows that the values $\Theta_j(t)$ approximating $\theta_j(t)$ will be the exact solution values of the system of the n ordinary differential equations

$$\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t} = rA\Theta(t), \tag{3}$$

where $\Theta(t) = [\Theta_1(t)\Theta_2(t)\cdots\Theta_n(t)]^{\mathrm{T}}$, $r = \nu/h^2$ and A is a tridiagonal matrix of order n as below

$$A = \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix}. \tag{4}$$

It is necessary to mention that the boundary conditions at points $x = 0$ and $x = 1$ have been written as

$$\frac{\Theta_1(t) - \Theta_0(t)}{h} = 0, \quad \frac{\Theta_{n+1}(t) - \Theta_n(t)}{h} = 0, \quad t > 0,$$

respectively. It can be easily seen that the solution of Equation (3) is

$$\Theta(t) = \exp(rtA)\Theta(0), \tag{5}$$

and if the solution is available at time level t then the solution at level $t + k$ is [26]

$$\Theta(t + k) = \exp(rkA)\Theta(t). \tag{6}$$

Hence, there are two usual ways for computing the solution of Equation (3) at time level t_f. One is to use Equation (5) substituting t by t_f. The other is to pick a natural number s, set $k = t_f/s$ and then use Equation (6) s times, recursively. If the solution is needed at some interior time levels then the second method is preferred to the first one.

LEMMA 2.1 *The eigenvalues and eigenvectors of the tridiagonal matrix* (4) *are given by*

$$\lambda_j = -2 + 2\cos\frac{(j-1)\pi}{n}, \quad j = 1, \ldots, n, \tag{7}$$

and

$$x_j = \begin{pmatrix} \cos(1j\pi/2n) \\ \cos(3j\pi/2n) \\ \cos(5j\pi/2n) \\ \vdots \\ \cos((2n-3)j\pi/2n) \end{pmatrix}, \quad j = 1, \ldots, n, \tag{8}$$

i.e., $Ax_j = \lambda_j x_j, \quad j = 1, \ldots, n$. *Moreover, the matrix* A *is diagonalizable and matrix* $P = (x_1\ x_2\ \cdots\ x_n)$ *diagonalizes* A, *i.e.,* $P^{-1}AP = D$, *where* $D = \mathrm{diag}(\lambda_1\ \lambda_2\ \cdots\ \lambda_n)$.

Proof See [24,34]. ∎

Obviously, the entries of P can be written as

$$p_{ij} = \cos\frac{(2i-1)(j-1)\pi}{2n}, \quad i, j = 1, \ldots, n.$$

By using Lemma 2.1 we have $A^j = PD^jP^{-1}$, $j = 1, 2, \ldots$. Hence

$$\exp(rkA) = P\exp(rkD)P^{-1}.$$

On the other hand $\exp(rkD) = \mathrm{diag}(\mathrm{e}^{rk\lambda_1}\mathrm{e}^{rk\lambda_2}\cdots\mathrm{e}^{rk\lambda_n})$. Therefore, it is enough to find an explicit expression for P^{-1}.

Hereafter, we denote the identity matrix of order n and the matrix of order n whose entries are all ones by I and U, respectively.

LEMMA 2.2 *By the above notations we have*

$$P^{-1} = P^{\mathrm{T}}R^{-1}, \tag{9}$$

where $R = (n/2)(I + (1/n)U)$.

Proof It is enough to show that $PP^{\mathrm{T}} = R$, *i.e.*,

$$p_i^{\mathrm{T}} p_j = \begin{cases} \dfrac{n+1}{2}, & i = j, \\ \dfrac{1}{2}, & i \neq j, \end{cases}$$

where p_k is the kth column of the matrix P. By using the well-known trigonometric identity

$$\sum_{k=1}^{n} \cos k\theta = \frac{\sin(n + (1/2))\theta}{2\sin(\theta/2)} - \frac{1}{2}, \tag{10}$$

and a little computation the desired relation can be obtained; the details are omitted. ■

LEMMA 2.3 *The inverse of the matrix R defined in Lemma 2.2 is given by*

$$R^{-1} = \frac{2}{n}\left(I - \frac{1}{2n}U\right). \tag{11}$$

Proof Let u be an n-vector whose entries are all ones. Then we have

$$R = \frac{n}{2}\left(I + \frac{1}{n}uu^{\mathrm{T}}\right).$$

Now by using the Sherman–Morrison formula [24]

$$(I + cd^{\mathrm{T}})^{-1} = I - \frac{cd^{\mathrm{T}}}{1 + d^{\mathrm{T}}c}, \quad \text{for } c, d \in \mathbb{R}^n, \text{ with } d^{\mathrm{T}}c \neq -1,$$

we deduce

$$R^{-1} = \frac{2}{n}\left(I - \frac{1}{2n}U\right).$$

■

LEMMA 2.4 *Let $P = (p_{ij})$ and $Q = P^{-1} = (q_{ij})$. Then*

$$q_{1j} = \frac{1}{n}, \quad j = 1, \ldots, n,$$
$$q_{ij} = \frac{2}{n} p_{ji}, \quad i = 2, \ldots, n, \; j = 1, \ldots, n.$$

Proof Again by using identity (10), it can be easily verified that

$$P^{\mathrm{T}}U = \begin{pmatrix} n & n & \cdots & n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Substituting this in Equation (11) yields the desired relation. ■

THEOREM 2.5 *Let A be the tridiagonal matrix defined in Equation* (4) *and* $Z = \exp(\alpha A) = (z_{ij})$. *Then*

$$z_{ij} = \frac{1}{n}\left(1 + 2\sum_{\ell=2}^{n} e^{\alpha\lambda_\ell}\cos(2i-1)z_\ell \cos(2j-1)z_\ell\right), \tag{12}$$

where $z_\ell = ((\ell-1)\pi)/2n$ *and* $\lambda_\ell = -2 + 2\cos z_\ell/2$.

Proof We have

$$\begin{aligned} z_{ij} &= (P e^{\alpha D} Q)_{ij} \\ &= \begin{pmatrix} p_{i1} & p_{i2} & \cdots & p_{in} \end{pmatrix} \begin{pmatrix} e^{\alpha\lambda_1} q_{1j} \\ e^{\alpha\lambda_2} q_{2j} \\ \vdots \\ e^{\alpha\lambda_n} q_{nj} \end{pmatrix} \\ &= \sum_{\ell=1}^{n} e^{\alpha\lambda_\ell} p_{i\ell} q_{\ell j}. \end{aligned}$$

On the other hand, we have $\lambda_1 = 0$, $p_{i1} = 1$ and $q_{1j} = 1/n$. Hence Lemma 2.4 yields

$$z_{ij} = \frac{1}{n}\left(1 + 2\sum_{\ell=2}^{n} e^{\alpha\lambda_\ell} p_{i\ell} p_{j\ell}\right),$$

which is the same relation as Equation (12). ■

By using this theorem, for any n-vector y, the explicit expression for the matrices $\exp(rkA)y$ and $\exp(rt_f A)y$ can be obtained by replacing α by rk and rt_f, respectively. Here we mention that all of the computations can be done in parallel.

For the method based on Equation (6), the stability can be verified as follows. The eigenvalues of the matrix $\exp(rkA)$ are given by $e^{rk\lambda_\ell}$, $\ell = 1, 2, \ldots, n$, where λ_ℓ is defined as

$$\lambda_\ell = -2 + 2\cos\frac{(\ell-1)\pi}{n} = -4\sin^2\frac{(\ell-1)\pi}{2n}.$$

Hence, $e^{rk\lambda_\ell} \le 1$, $\ell = 1, 2, \ldots, n$. Therefore, the method is unconditionally stable.

3. Numerical results

In this section, we give the numerical experiments of applying the method based on Equation (5) for solving the two problems considered in Section 1. All of the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer Pentium 3 – 800EB MHz. It is necessary to mention that after computing the approximate solution of Equation (3), we compute the approximate solution of the Burgers's equation by

$$U(x_i, t) = -\frac{\nu}{h}\left(\frac{\Theta(x_{i+1}, t) - \Theta(x_{i-1}, t)}{\Theta(x_i, t)}\right), \quad i = 1, 2, \ldots, n.$$

Numerical results for $\nu = 1$ and $\nu = 0.1$ with $h = 0.005$ at time levels $t_f = 0.2$, 0.4 and 0.6 for $x = 0.25$, 0.5 and 0.75 have been given in Tables 1 and 2, respectively. As we observe, the approximate solutions of the two problems are in good agreement with the exact solution.

Table 1. Numerical results for $\nu = 1$ and $h = 0.005$ at time levels $t_f = 0.2$, 0.4 and 0.6 for $x = 0.25$, 0.5 and 0.75.

		Problem 1		Problem 2	
	x	Approximate	Exact	Approximate	Exact
$t_f = 0.2$	0.25	0.09443	0.09644	0.09738	0.09947
	0.50	0.13609	0.13847	0.14040	0.14289
	0.75	0.09731	0.09944	0.10044	0.10266
$t_f = 0.4$	0.25	0.01303	0.01357	0.01344	0.01400
	0.50	0.01853	0.01924	0.01912	0.01985
	0.75	0.01308	0.01363	0.01350	0.01407
$t_f = 0.6$	0.25	0.00178	0.00189	0.00183	0.00195
	0.50	0.00252	0.00267	0.00260	0.00276
	0.75	0.00178	0.00189	0.00183	0.00195

Table 2. Numerical results for $\nu = 0.1$ and $h = 0.005$ at time levels $t_f = 0.2$, 0.4 and 0.6 for $x = 0.25$, 0.5 and 0.75.

		Problem 1		Problem 2	
	x	Approximate	Exact	Approximate	Exact
$t_f = 0.2$	0.25	0.42776	0.42932	0.44507	0.44682
	0.50	0.75355	0.75381	0.77278	0.77311
	0.75	0.74793	0.74914	0.77214	0.77345
$t_f = 0.4$	0.25	0.30704	0.30889	0.31554	0.31753
	0.50	0.56867	0.56963	0.58348	0.58454
	0.75	0.62284	0.62544	0.64289	0.64562
$t_f = 0.6$	0.25	0.23898	0.24074	0.24429	0.24614
	0.50	0.44578	0.44721	0.45646	0.45798
	0.75	0.48380	0.48721	0.49911	0.50268

In Tables 3 and 4, the approximate solutions of Problems 1 and 2 at time level $t_f = 0.01$ with $\nu = 10$ and for different values of grid points and stepsizes are given. These tables show the effect of decrease in the stepsize on the convergence. In fact, they show that the error in both solutions decreases as h decreases.

In Tables 5 and 6, we compare the numerical results of our method (NM) with the method proposed by Kutluay et al. (KED) in [17]. To do so, we assume $k = 0.0001$ for the KED method

Table 3. Numerical results for Problem 1 with $\nu = 10$ and $t_f = 0.01$ for different values of grid points.

x	$h = 1/100$	$h = 1/200$	$h = 1/300$	$h = 1/400$	$h = 1/500$	Exact
0.1	0.10848	0.11155	0.11257	0.11308	0.11339	0.11461
0.2	0.21207	0.21513	0.21615	0.21665	0.21696	0.21816
0.3	0.29473	0.29770	0.29867	0.29916	0.29945	0.30062
0.4	0.34822	0.35108	0.35203	0.35250	0.35278	0.35390
0.5	0.36709	0.36992	0.37085	0.37132	0.37159	0.37270
0.6	0.34931	0.35219	0.35314	0.35361	0.35389	0.35502
0.7	0.29649	0.29948	0.30047	0.30096	0.30125	0.30243
0.8	0.21381	0.21691	0.21793	0.21844	0.21875	0.21997
0.9	0.10952	0.11263	0.11367	0.11418	0.11449	0.11573

Table 4. Numerical results for Problem 2 with $\nu = 10$ and $t_f = 0.01$ for different values of grid points.

x	$h = 1/100$	$h = 1/200$	$h = 1/300$	$h = 1/400$	$h = 1/500$	Exact
0.1	0.11191	0.11510	0.11616	0.11669	0.11700	0.11828
0.2	0.21877	0.22197	0.22303	0.22356	0.22387	0.22514
0.3	0.30405	0.30716	0.30819	0.30870	0.30901	0.31023
0.4	0.35924	0.36226	0.36325	0.36375	0.36404	0.36522
0.5	0.37872	0.38171	0.38269	0.38318	0.38347	0.38464
0.6	0.36040	0.36343	0.36443	0.36492	0.36522	0.36641
0.7	0.30592	0.30905	0.31009	0.31061	0.31091	0.31215
0.8	0.22062	0.22385	0.22492	0.22546	0.22578	0.22706
0.9	0.11301	0.11624	0.11732	0.11785	0.11817	0.11946

Table 5. Numerical results for $\nu = 0.1$ and $k = 0.0001$ at different time levels (t_f) with $h = 0.0125$ for Problem 1.

x	t_f	Exact	NM	KED
0.25	0.4	0.30889	0.30415	0.31215
	0.6	0.24074	0.23629	0.24360
	0.8	0.19568	0.19150	0.19815
	1.0	0.16256	0.15861	0.16473
0.50	0.4	0.56963	0.56711	0.57293
	0.6	0.44721	0.44360	0.45088
	0.8	0.35924	0.35486	0.36286
	1.0	0.29192	0.28710	0.29532
0.75	0.4	0.62544	0.61874	0.63038
	0.6	0.48721	0.47855	0.49268
	0.8	0.37392	0.36467	0.37912
	1.0	0.28747	0.27860	0.29204

and for both methods $h = 0.0125$. The approximate solutions at different time levels and grid points are given. As we see, the numerical results of both methods are comparable.

For $\nu = 0.01$, the computed solution of the problems by the method presented in this paper has been displayed in Figure 1. Here, we assume $h = 0.005$ and the solutions for different time levels are shown (Problem 1, top and Problem 2, bottom). This figure is in agreement with the physical behaviour of the problem.

Table 6. Numerical results for $\nu = 0.1$ and $k = 0.0001$ at different time levels (t_f) with $h = 0.0125$ for Problem 2.

x	t_f	Exact	NM	KED
0.25	0.4	0.31753	0.31247	0.32091
	0.6	0.24614	0.24148	0.24910
	0.8	0.19956	0.19524	0.20211
	1.0	0.16560	0.16153	0.16782
0.50	0.4	0.58454	0.58176	0.58788
	0.6	0.45798	0.45414	0.46174
	0.8	0.36740	0.36283	0.37111
	1.0	0.29835	0.29336	0.30183
0.75	0.4	0.64562	0.63858	0.65054
	0.6	0.50268	0.49362	0.50825
	0.8	0.38534	0.37570	0.39068
	1.0	0.29586	0.28663	0.30057

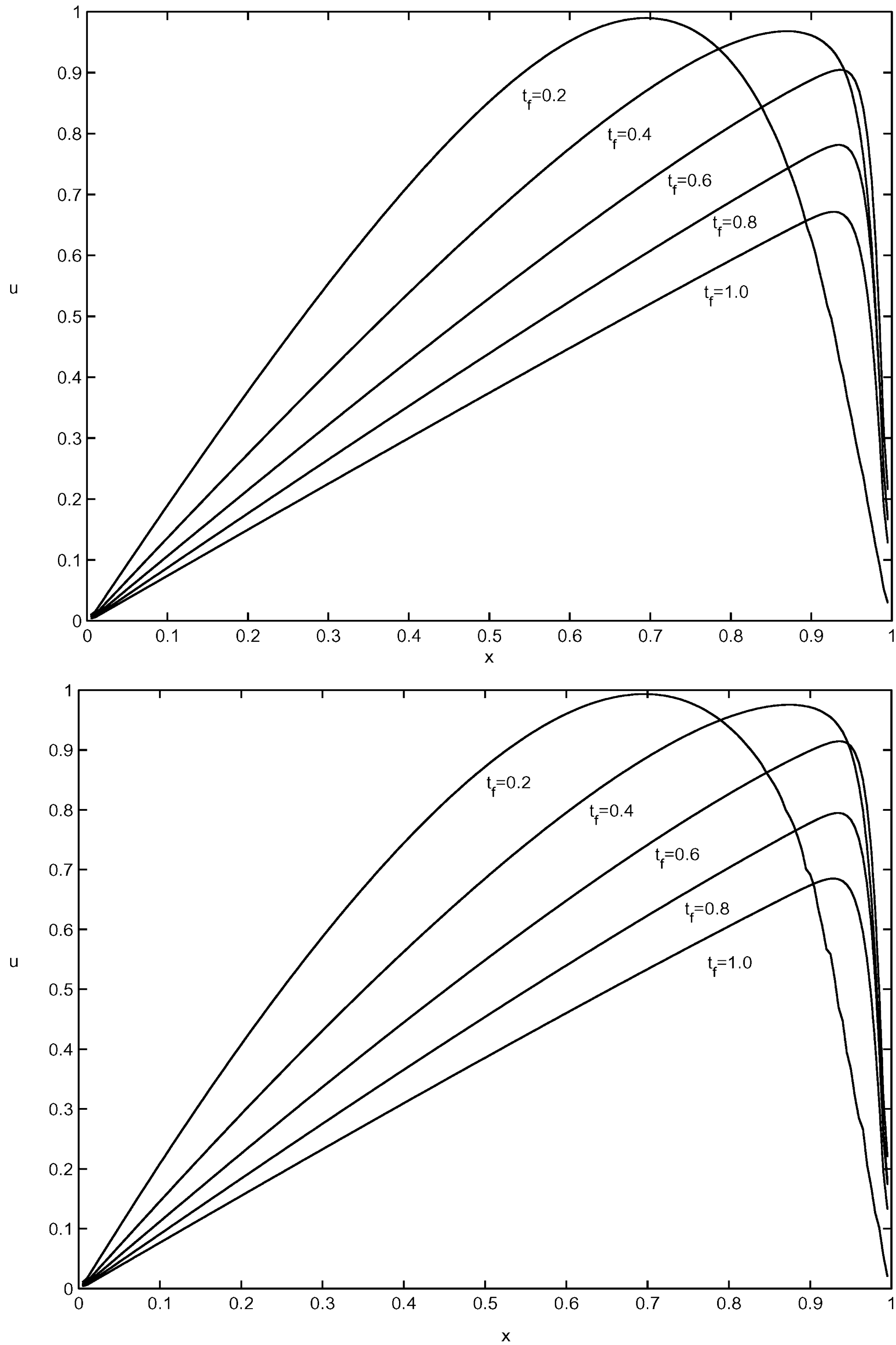

Figure 1. Approximate solutions of the problems at different time levels with $\nu = 0.01$ and $h = 0.005$ (Problem 1, top and Problem 2, bottom).

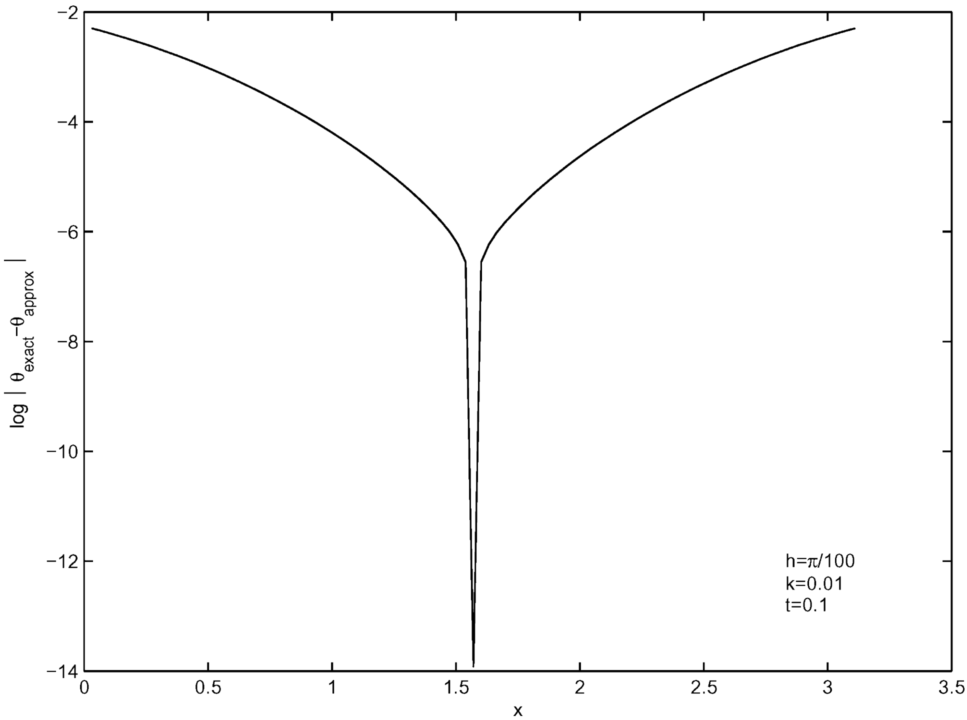

Figure 2. Logarithm of the absolute errors at time $t = 0.1$ for Equation (13).

Since the method described in this paper is directly applied for solving the heat equation, we give the numerical results of implementing the method for solving the following model

$$\begin{aligned} \theta_t &= \theta_{xx}, \quad 0 < x < \pi, \ \ t > 0, \\ \theta(x,0) &= \cos x, \quad 0 < x < \pi, \\ \theta_x(0,t) &= \theta_x(\pi,t) = 0, \quad t > 0. \end{aligned} \tag{13}$$

The analytical solution of this model is $\theta(x,t) = \mathrm{e}^{-t}\cos x$. The logarithm of the absolute errors at time level $t = 0.1$ with $h = \pi/100$ and $k = 0.01$ are displayed in Figure 2. The figure shows the efficiency of the method.

4. Conclusion

In this paper, the heat equation arisen from the Burgers's equation by the Hopf–Cole transformation is semi-discretized. This kind of discretization results in a large system of ordinary differential equations. We have presented an efficient method for solving this system. Numerical examples confirm the efficiency of the method.

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References

[1] R. Bahadir and M. Saglam, *A mixed finite difference and boundary element approach to one-dimensional Burgers's equation*, Appl. Math. Comput. 160 (2005), pp. 663–673.

[2] J. Caldwell, P. Wanless, and A.E. Cook, *A finite element approach to Burgers's equation*, Appl. Math. Modelling 5 (1981), pp. 189–193.

[3] J. Caldwell and P. Smith, *Solution of Burgers' equation with a large Reynolds number*, Appl. Math. Model 6 (1982), pp. 381–385.

[4] S.A. Elwakil et al., *Modified extended tanh-function method for solving nonlinear partial differential equations*, Phys. Lett. A 299 (2002), pp. 179–188.

[5] ———, *Modified extended tanh-function method and its applications to nonlinear equations*, Appl. Math. Comput. 161 (2005), pp. 403–412.

[6] F. Engui and Z. Hongqing, *Backlund transformation and exact solutions for Whitham–Broer–Kaup equations in shallow water*, Appl. Math. Mech. 19 (1998), pp. 713–716.

[7] Z. Fu et al., *New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations*, Phys. Lett. A 290 (2001) pp. 72–76.

[8] A. Gorguis, *A comparison between Cole–Hopf transformation and decomposition method for solving Burgers' equations*, Appl. Math. Comput. 173 (2006), pp. 126–136.

[9] M. Gülsu, *A finite difference approach for solution of Buregers' equation*, Appl. Math. Comput. 175 (2006), pp. 1245–1255.

[10] M. Gülsu and T. Özis, *Numerical solution of Burgers' equation with restrictive Taylor approximation*, Appl. Math. Comput. 171 (2005), pp. 1192–1200.

[11] Y.C. Hon and X.Z. Mao, *An efficient numerical schme for Burgers' equation*, Appl. Math. Comput. 95 (1998), pp. 37–50.

[12] E. Hopf, *The partial differential equation* $U_t + UU_x = \mu U_{xx}$, Commun. Pure Appl. Math. 3 (1950), pp. 201–230.

[13] H.N.A. Ismail and E.M.E. Elbarbary, *Restrictive Taylor's approximation and parabolic partial differential equations*, Int. J. Comput. Math. 78 (2001), pp. 73–82.

[14] K. Konno and M. Wadati, *Simple derivation of Bäcklund transformation from Riccati form of inverse method*, Prog. Theor. Phys. 53 (1978), pp. 1652–1656.

[15] S. Kutluay and A. Esen, *A lumped Galerkin method for solving the Burgers equation*, Int. J. Comput. Math. 81 (2004), pp. 1433–1444.

[16] S. Kutluay, A.R. Bahadir, and A. Özdes, *Numerical solution of one-dimensional Burgers equation: explicit and exact-explicit finite difference methods*, J. Comput. Appl. Math. 103 (1999), pp. 251–261.

[17] S. Kutluay, A. Esen, and I. Dag, *Numerical solutions of the Burgers' equation by the least-squares quadratic B-spline finite element method*, J. Comput. Appl. Math. 167 (2004), pp. 21–33.

[18] D. Li and H.Q. Zhang, *A further extended tanh-function method and new solitonlike solutions to the integrable Broer-Kaup (BK) equations in* $(2 + 1)$ *dimemional space*, Appl. Math. Comput. 147 (2004), pp. 537–545.

[19] S. Liu et al., *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations*, Phys. Lett. A 289 (2001), pp. 69—74.

[20] Z.S. Lü, *An explicit Bäcklund transformation of Burgers equation with applications*, Commun. Theor. Phys. 44 (2005), pp. 987–989.

[21] W. Malfliet, *Solitary wave solutions of nonlinear wave equations*, Am. J. Phys. 60 (1992), pp. 650–654.

[22] W. Malfliet and W. Hereman, *The tanh method: II. Perturbation technique for conservative systems*, Phys. Scr. 54 (1996), pp. 569–575.

[23] V.A. Matveev and M.A. Salle, *Darboux Transformation and Solitons*, Springer-Verlag, Berlin, Heidelberg, 1991.

[24] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, 2004.

[25] E.J. Parkes and B.R. Duffy, *An automated tanh-function method for finding solitary wave solutions to non-linear evolution equations*, Comput. Phys. Commun. 98 (1996), pp. 288–300.

[26] G.D. Smith, *Numerical Solution of Partial Differential Equations (Finite Difference methods)*, Oxford University Press, New York, 1985.

[27] A.A. Soliman, *The modified extended tanh-function method for solving Burgers-type equations*, Physica A 361 (2006), pp. 394–404.

[28] J.W. Thomas, *Numerical Partial Diferential Equations: Finite Difference Methods*, Springer-Verlag, New York, 1995.

[29] M. Wadati, H. Sanuki, and K. Konno, *Relationships among inverse method, bäcklund transformation and an infinite number of conservation laws*, Prog. Theor. Phys. 53 (1975), pp. 419–436.

[30] A.M. Wazwaz, *A reliable treatment of the physical structure for the nonlinear equation K(m, n)*, Appl. Math. Comput. 163 (2005), pp. 1081–1095.

[31] ———, *Nonlinear dispersive special type of the Zakharov–Kuznetsov equation ZK(n, n) with compact and noncompact structures*, Appl. Math. Comput. 161 (2005), pp. 577–590.

[32] ———, *Exact solutions with compact and noncompact structures for the onedimensional generalized Benjamin–Bona–Mahony equation*, Commun. Nonlinear Sci. Numer. Simul. 10 (2005), pp. 855–867.

[33] ———, *The tanh and the sine–cosine methods for a reliable treatment of the modified equal width equation and its variants*, Commun. Nonlinear Sci. Numer. Simul. 11 (2006), pp. 148–160.

[34] W.C. Yueh, *Eigenvalues of several tridiagonal matrices*, Appl. Math. E-Notes 5 (2005), pp. 66–74.